Accuracy

Real Part of Dielectric Constant

Measurements of the rod diameter were made with a bench micrometer. All samples had some taper along the length and some ellipticity about the cross section. It is estimated that the effective diameter was measured with an accuracy of $\frac{1}{2}$ per cent. This would contribute an error of 1 per cent in the measurement of K'.

The resonant frequency was determined by taking the average of the two frequencies at which the output was a given fraction below the resonant output. Since all frequency measurements were made using a frequency counter, this method is very precise. A maximum error of 2 per cent was attributed to the electrical measurement on the basis of spread in results using different cavities. The maximum over-all error in measuring K' was thus 3 per cent.

Tan δ

As noted above, the loss tangent of the cavity was determined by use of (5). The measurement was made twice using a different value of α each time. If the two different determinations disagreed by more than 3 per cent additional measurements were made. A frequency counter was used in making the measurements. The maximum over-all error in tan δ was taken as 0.0005.

General Synthesis of Asymmetric Multi-Element Coupled-Transmission-Line Directional Couplers*

RALPH LEVY†

Summary—An exact synthesis procedure is derived for a class of asymmetric multi-element coupled-transmission-line directional couplers with any number of elements. It is based on the equivalence between the theory of the directional coupler and that of a stepped quarter-wavelength filter. This can be treated using Richards' theorem for the synthesis of transmission-line distributed networks, as described previously by Riblet. The method is extended to give a general expression for the input reflection coefficient of the stepped filter, which corresponds to the voltage coupling of the directional coupler. Explicit formulas for the parameters of two, three, four and five couplers are derived and the extension to larger number of elements is straightforward. Two and three element couplers have been designed on this theoretical basis, and show excellent agreement with theory, for example a three element coupler of 20 db ±0.5 db over a 6:1 bandwidth, and a two element coupler of 3.2 db ±0.85 db over a 6.7:1 bandwidth. It is possible to design a 3-db ±0.43 db-coupler for decade bandwidths using only four elements. The 3 db-couplers may be used as 90° hybrids by careful choice of reference planes in the output parts.

I. Introduction

Coupled-transmission-line directional couplers have been described by a number of authors (Oliver,[1] Jones and Bolljahn,[2] and Shimizu and Jones[3]). The simple quarter-wavelength directional coupler has perfect isolation and perfect input match, but the coupling varies according to

$$L = 1 + \tfrac{1}{4}(Z_{oe} - 1/Z_{oe})^2 \sin^2 \theta \tag{1}$$

where L is the power insertion loss from the input to arm 4 (Fig. 1). Z_{oe} and Z_{oo} are the impedances of the even and odd modes in the coupled line normalized to the impedance of the input lines, and are related by

$$Z_{oe} Z_{oo} = 1. \tag{2}$$

Eq. (1) indicates that a useful bandwidth of rather more than one octave is obtained from the simple quarter-wavelength section. Shimizu and Jones[3] have described how much greater bandwidths may be obtained by cascading three coupled-line sections to form a three-quarter wavelength coupler. Considerable simplification of their equation (10) for the coupling is possible (see Appendix I), leading to the formula

$$L = 1 + \frac{1}{4}\left[\left\{ 2\left(Z_{oe}' - \frac{1}{Z_{oe}'}\right) + \left(Z_{oe} - \frac{1}{Z_{oe}}\right)\right\} \cdot \sin\theta \cos^2\theta - \left(\frac{Z_{oe}'^2}{Z_{oe}} - \frac{Z_{oe}}{Z_{oe}'^2}\right)\sin^3\theta \right]^2 \tag{3}$$

where, as in (1), L is the insertion loss from the input

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[1] B. M. Oliver, "Directional electromagnetic couplers," Proc. IRE, vol. 42, pp. 1688–1692; November, 1954.

[2] E. M. T. Jones and J. T. Bolljahn, "Coupled-strip transmission line filters and directional couplers," IRE Trans. on Microwave Theory and Techniques, vol. MTT-4, pp. 75–81; April, 1956.

[3] J. K. Shimizu and E. M. T. Jones, "Coupled-transmission-line directional couplers," IRE Trans. on Microwave Theory and Techniques, vol. MTT-6, pp. 403–410; October, 1958.

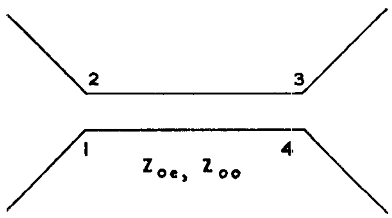

Fig. 1—Quarter-wavelength directional coupler.

to arm 4. This equation can be analyzed quite readily for any coupling and bandwidth required, *e.g.*, for a coupling of 3 db ±0.4 db the bandwidth is 5:1. In spite of this large bandwidth the three-quarter wavelength coupler is not optimum in the sense of having maximum bandwidth for a given coupling tolerance. Eq. (3) is a cubic in $\sin^2\theta$ or $\cos^2\theta$ and the coupling characteristic should have two equal ripples to be of optimum form. In fact it has only one ripple. The reason for this is the restriction placed on the form of coupling by making the two outer elements equal. When the three elements are allowed to be all unequal, then it is possible to obtain two ripples in the coupling characteristic. Fel'dshtein[4] has shown that for such an n-element asymmetric coupler the power division between arms 2 and 4 should take the form

$$\eta = \frac{|S_{12}|^2}{|S_{14}|^2} = \beta^2 - h^2 T_n^2\left(\frac{\cos\theta}{\cos\theta_0}\right) \tag{4}$$

where

$$1/\cos\theta_0 = \cosh\left(\frac{1}{n}\cosh^{-1}\beta/h\right) \tag{5}$$

and the pass band extends from θ_0 to $\pi-\theta_0$. The coupling or insertion loss to arm 4 is then

$$L = \frac{1}{|S_{14}|^2} = 1 + \beta^2 - h^2 T_n^2\left(\frac{\cos\theta}{\cos\theta_0}\right). \tag{6}$$

T_n denotes the Chebyshev function of the first kind of degree n.

A key to the solution of the synthesis problem is found by noting that the couplings to arms 2 and 4 are respectively the reflection and transmission coefficients of a cascaded set of transmission lines of electrical length θ with the impedances $Z_1, Z_2, \cdots, Z_n$ (where as an abbreviation Z_r is written for Z_{oer}) terminated by lines of unit impedance. This equivalence, which is illustrated in Fig. 2, has been proved in current papers,[5,6] but it is

[4] A. L. Fel'dshtein, "Synthesis of stepped directional couplers," *Radiotekh. i Electron.*, vol. 6, pp. 234–240; February, 1961.

[5] R. Levy, "Coupled Transmission Line Theory and the Design of Ultra-Broadband Microwave Components," presented at IEE Conference on Components for Microwave Circuits, London, England; September 1962.

[6] L. Young, "The analytical equivalence of T.E.M. mode directional couplers and transmission-line-stepped-impedance filters," PROC. I.E.E., vol. 110, pp. 275–281; February, 1963. Leo Young has informed the author that he has independently reached the conclusion that the analysis of T.E.M. mode couplers is equivalent to that of quarter-wave stepped transformers, and has heard of similar work by S. B. Cohn and H. J. Riblet.

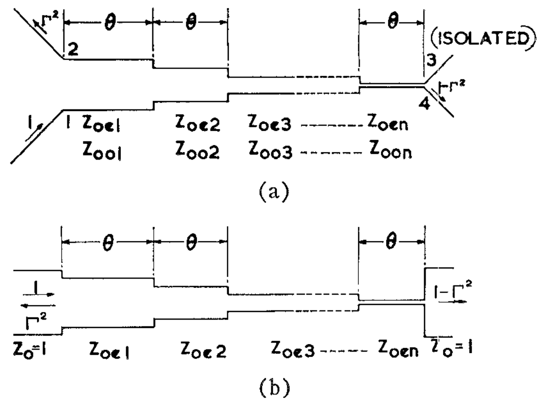

Fig. 2—(a) Equivalence between an n-element coupler (b) and n stepped quarter-wavelength lines.

presented in Appendix I for completeness. Its application leads immediately to the simple derivation of equations such as (1) and (3). In the general lossless case of n elements the transfer matrix is given by

$$\prod_{r=1}^{n}\begin{bmatrix}\cos\theta & jZ_r\sin\theta \\ j/Z_r\sin\theta & \cos\theta\end{bmatrix} = \begin{bmatrix}A_n & jB_n \\ jC_n & D_n\end{bmatrix} \tag{7}$$

and the insertion loss to arm 4 of the coupler is the insertion loss of the two-port network, *i.e.*,

$$L = 1 + \tfrac{1}{4}(A_n - D_n)^2 + \tfrac{1}{4}(B_n - C_n)^2. \tag{8}$$

Since (8) can be expressed as a polynomial in powers of $\cos^2\theta$ of degree n, the synthesis of the optimum coupling characteristic may be performed by equating coefficients of $\cos^{2r}\theta$ in (8) with corresponding coefficients in (6). This leads to n simultaneous equations in the n variables $Z_1, Z_2, \cdots, Z_n$, which may be solved for values of n up to $n=3$. Beyond this value of n the process rapidly becomes untractable. A solution of the problem is available, however, by recourse to modern synthesis techniques.

II. Synthesis Procedure

The synthesis is performed using a theorem due to Richards[7] and extended by Riblet[8] to the synthesis of quarter-wave impedance transformers. The present problem is similar to that treated by Riblet, but before applying the techniques, it is first necessary to extend his theorem to the synthesis of any transmission line stepped-impedance filter, as described in Appendix II. It follows that (6) can always be synthesized as a stepped impedance filter with real positive characteristic impedances. Introducing the propagation function γ, given by

$$\gamma = \sigma + j\theta \tag{9}$$

[7] P. I. Richards, "Resistor-transmission-line circuits," PROC. IRE, vol. 34, pp. 217–220; September, 1948.

[8] H. J. Riblet, "General synthesis of quarter-wave impedance transformers," IRE TRANS. ON MICROWAVE THEORY AND TECHNIQUES, vol. MTT-5, pp. 36–43; January, 1957.

where θ is the electrical length of the uniform line, and σ is the attenuation coefficient, then Richards showed that the frequency transformation

$$t = \tanh \gamma \tag{10}$$

maps the complex impedance plane $Z(\gamma)$ into the plane $Z(t)$, where t is a new complex variable equivalent to the complex frequency variable in ordinary lumped-element synthesis theory. Thus $Z(t)$ satisfies Brune's condition that it must be a positive real function if it is to represent a realizable impedance. For this condition to hold the reflection coefficient $\Gamma(t)$ must be a function which is regular (*i.e.*, with no poles) in the right-half plane. In the case of minimum phase networks there is the additional condition that $\Gamma(t)$ has no zeros in the right-half plane.

The required insertion loss function is given by (6), so that the square of the reflection coefficient is

$$\left|\Gamma\left(\frac{\cos\theta}{\cos\theta_0}\right)\right|^2 = \frac{L-1}{L} = \frac{\beta^2 - h^2 T_n^2\left(\frac{\cos\theta}{\cos\theta_0}\right)}{1+\beta^2 - h^2 T_n\left(\frac{\cos\theta}{\cos\theta_0}\right)}. \tag{11}$$

$|\Gamma(\cos\theta/\cos\theta_0)|^2$ represents the power reflection coefficient of a lossless network, and it is necessary to generalize (11) to give $|\Gamma(t)|^2$. The generalization will reduce to (11) for real frequencies (*i.e.*, for $t = \tanh j\theta = j\tan\theta$). The roots θ, of the numerator in (11) are given by

$$\cosh\left(n \cosh^{-1}\frac{\cos\theta_r}{\cos\theta_0}\right) = \pm\frac{\beta}{h}$$

i.e.,

$$n\cosh^{-1}\frac{\cos\theta_r}{\cos\theta_0} = \cosh^{-1}\frac{\beta}{h} + jr\pi$$

$$r = 1, 2, \cdots, n).$$

Using (5) this becomes

$$\cos\theta_r = \frac{\cosh\left(1/n\cosh^{-1}\beta/h + jr\pi/n\right)}{\cosh\left(1/n\cosh^{-1}\beta/h\right)}. \tag{12}$$

$$r = 1, 2, \cdots, n)$$

This may now be written in terms of $\tan^2\theta_r$ and hence generalized to give the complex roots

$$t_r^2 = 1 - \frac{\cosh^2 J/n}{\cosh^2\left(J/n + jr\pi/n\right)} \tag{13}$$

where

$$J = \cosh^{-1}\beta/h. \tag{14}$$

After some trigonometrical manipulation (13) can be written in terms of its real and imaginary parts as

$$t_r^2 = 1 - \frac{\cosh^2 J/n(\cos^2 r\pi/n\cosh^2 J/n - \sin^2 r\pi/n\sinh^2 J/n)}{(\cosh^2 J/n - \sin^2 r\pi/n)^2} + j\frac{\cosh^2 J/n\sin 2r\pi/n\sinh 2J/n}{2(\cosh^2 J/n - \sin^2 r\pi/n)^2}. \tag{15}$$

Similarly the roots of the denominator in (11) are given by

$$t_r'^2 = 1 - \frac{\cosh^2 J/n(\cos^2 r\pi/n\cosh^2 H/n - \sin^2 r\pi/n\sinh^2 H/n)}{(\cosh^2 H/n - \sin^2 r\pi/n)^2} + j\frac{\cosh^2 J/n\sin 2r\pi/n\sinh 2H/n}{2(\cosh^2 H/n - \sin^2 r\pi/n)^2} \tag{16}$$

where

$$H = \cosh^{-1}\sqrt{1+\beta^2}/h. \tag{17}$$

The expression for $|\Gamma(t)|^2$ finally takes the form

$$|\Gamma(t)|^2 = K\frac{(t^2 - t_1^2)(t^2 - t_2^2)\cdots(t^2 - t_n^2)}{(t^2 - t_1'^2)(t^2 - t_2'^2)\cdots(t^2 - t_n'^2)} \tag{18}$$

$$K = \begin{cases} \dfrac{\beta^2}{1+\beta^2} = \left(\dfrac{\cosh J}{\cosh H}\right)^2 \ (n \text{ odd}) & (19) \\ \dfrac{\beta^2 - h^2}{1+\beta^2 - h^2} = \left(\dfrac{\sinh J}{\sinh H}\right)^2 \ (n \text{ even}). & (20) \end{cases}$$

The values of the roots t_r^2, $t_r'^2$ given by (15) and (16) are either real, or occur in complex conjugate pairs. For example when n is odd there is a real root for $r = n$, and the others occur in complex conjugate pairs $(r, n-r)$ for $r = 1, 2, \cdots, \frac{1}{2}(n-1)$. Eq. (18) becomes, for n odd,

$$|\Gamma(t)|^2 = \frac{\beta^2}{(1+\beta^2)}\frac{(t^2 - t_n^2)}{(t^2 - t_n'^2)}\prod_{r=1}^{\frac{1}{2}(n-1)}\frac{(t^2 - t_r^2)(t^2 - t_r^{2*})}{(t^2 - t_r'^2)(t^2 - t_r'^{2*})}. \tag{21}$$

As previously stated, $\Gamma(t)$ must have no poles or zeros in the right half plane, leading to the formula

$$\Gamma(t) = \frac{\beta}{\sqrt{1+\beta^2}}\frac{(t+t_n)}{(t+t_n')}\prod_{r=1}^{\frac{1}{2}(n-1)}\frac{(t+t_r)(t+t_r^*)}{(t+t_r')(t+t_r'^*)} \tag{22}$$

where the real parts of the t_r, t_r' are all positive. Eq. (22) is further simplified by forming the product

$$(t + t_r')(t + t_r'^*) = t^2 + (t_r' + t_r'^*)t + |t_r'|^2$$

and deriving expressions for the coefficients of this quadratic in t from (16), *i.e.*,

$$T_r' = (t_r' + t_r'^*) = \sqrt{2\left[1 - \frac{2\cosh^2 J/n(\cosh 2H/n \cos 2r\pi/n + 1)}{(\cosh 2H/n + \cos 2r\pi/n)^2} + |t_r'|^2\right]} \tag{23}$$

$$|t_r'|^2 = \frac{\sqrt{(\cosh 2(H+J)/n - \cos 2r\pi/n)(\cosh 2(H-J)/n - \cos 2r\pi/n)}}{(\cosh 2H/n + \cos 2r\pi/n)}. \tag{24}$$

Similar expressions for $T_r = (t_r + t_r^*)$ and $|t_r|^2$ are obtained by substituting J for H in (23) and (24). All factors in (22) now have positive real coefficients, complex numbers having been eliminated.

The generalized reflection coefficient for n even is similarly

$$\Gamma(t) = \frac{\sqrt{\beta^2 - h^2}(t + t_{n/2})(t + t_n)}{\sqrt{1 + \beta^2 - h^2}(t + t_{n/2}')(t + t_n')} \cdot \prod_{r=1}^{\frac{1}{2}n-1} \frac{[t^2 + (t_r + t_r^*)t + |t_r|^2]}{[t^2 + (t_r' + t_r'^*)t + |t_r'|^2]}. \tag{25}$$

Eqs. (22) or (25) express $\Gamma(t)$ as the quotient of two positive real polynomials of the nth degree, and from them the input impedance $Z(t)$ is derived from the formula

$$Z(t) = \frac{1 + \Gamma(t)}{1 - \Gamma(t)} = \frac{P_n(t)}{Q_n(t)} \tag{26}$$

where $P_n(t)$ and $Q_n(t)$ are positive real polynomials of the nth degree.

The next step in the synthesis is to find the values of the impedances of the n lines of electrical length θ, terminated by lines of unit impedance (Fig. 2) which gives this value of $Z(t)$. In terms of the propagation coefficient γ [(9)] the transfer matrix of a single line element of impedance Z is

$$\begin{bmatrix} \cosh\gamma & Z\sinh\gamma \\ 1/Z\sinh\gamma & \cosh\gamma \end{bmatrix} = \frac{1}{\sqrt{1-t^2}}\begin{bmatrix} 1 & Zt \\ t/Z & 1 \end{bmatrix} \tag{27}$$

where

$$t = \tanh\gamma.$$

Hence, the over-all transfer matrix for n elements is

$$\frac{1}{(1-t^2)^{n/2}}\prod_{r=1}^{n}\begin{bmatrix} 1 & Z_r t \\ t/Z_r & r \end{bmatrix} = \frac{1}{(1-t^2)^{n/2}}\begin{bmatrix} A(t) & B(t) \\ C(t) & D(t) \end{bmatrix}. \tag{28}$$

The input impedance is given by

$$Z(t) = \frac{A(t) + B(t)}{D(t) + C(t)} \tag{29}$$

where $A(t)$, $D(t)$ are even and $B(t)$, $C(t)$ are odd polynomials in t.

Now if (26) is written

$$Z(t) = \frac{EP_n(t) + OP_n(t)}{EQ_n(t) + OQ_n(t)} \tag{30}$$

where E and O refer to the even and odd parts of the polynomials $P_n(t)$ and $Q_n(t)$, it is clear that the over-all transfer matrix is

$$\frac{1}{(1-t^2)^{n/2}}\begin{bmatrix} EP_n(t) & OP_n(t) \\ OQ_n(t) & EQ_n(t) \end{bmatrix}. \tag{31}$$

This is the product of n matrices similar to that of (27), *i.e.*,

$$\begin{bmatrix} EP_n(t) & OP_n(t) \\ OQ_n(t) & EQ_n(t) \end{bmatrix} = \prod_{r=1}^{n}\begin{bmatrix} 1 & Z_r t \\ t/Z_r & 1 \end{bmatrix}. \tag{32}$$

In order to express the over-all matrix (32) as the product of its constituent matrices, it is premultiplied by the inverse matrix

$$\frac{1}{1-t^2} = \begin{bmatrix} 1 & -Z_1 t \\ -t/Z_1 & 1 \end{bmatrix} \tag{33}$$

and the value of Z_1 is obtained from the condition that all elements of the resulting matrix must be divisible by $(1-t^2)$. This process is repeated to give all impedances Z_r as a function of the original variables β and h of (6).

Example n=2: As an example of the synthesis procedure it is useful and instructive to derive the impedances for a two-element coupler. This is a simple but important case; for example, one can achieve a bandwidth of 5:1 for a coupling of 3 db±0.5 db, or a bandwidth of 4:1 for a coupling of 10 db±0.5 db.

With $n=2$ the roots given by (15) and (16) are all real, reducing to

$$t_1^2 = \frac{\cosh^2 J/2 + \sinh^2 J/2}{\sinh^2 J/2} = \frac{2\beta}{\beta - h}$$

$$t_2^2 = 0$$

$$t_1'^2 = \frac{\sinh^2 H/2 + \cosh^2 J/2}{\sinh^2 H/2} = \frac{\sqrt{1+\beta^2} + \beta}{\sqrt{1+\beta^2} - h}$$

$$t_2'^2 = \frac{\cosh^2 H/2 - \cosh^2 J/2}{\cosh^2 H/2} = \frac{\sqrt{1+\beta^2} - \beta}{\sqrt{1+\beta^2} + h}. \tag{34}$$

Applying (25) the reflection coefficient is

$$\Gamma(t) = \frac{\sqrt{\beta^2 + h^2}}{\sqrt{1 + \beta^2 - h^2}} \frac{t\left(t + \sqrt{\dfrac{2\beta}{\beta - h}}\right)}{\left(t + \dfrac{\sqrt{1+\beta^2}+\beta}{\sqrt{1+\beta^2}-h}\right)\left(t + \dfrac{\sqrt{1+\beta^2}-\beta}{\sqrt{1+\beta^2}+h}\right)}$$

$$= \frac{\sqrt{\beta^2 - h^2}\, t^2 + \sqrt{2\beta(\beta+h)}\, t}{\sqrt{1+\beta^2-h^2}\, t^2 + \left(\sqrt{(1+\beta^2) + (\beta+h)\sqrt{1+\beta^2} + \beta h} + \sqrt{(1+\beta^2) - (\beta+h)\sqrt{1+\beta^2} + \beta h}\right) t + 1} \cdot \quad (35)$$

Eqs. (26) and (31) give the transfer matrix

$$\frac{1}{1-t^2}\begin{vmatrix} at^2+1 & bt \\ ct & dt^2+1 \end{vmatrix} \quad (36)$$

where

$$a = \sqrt{1+\beta^2-h^2} + \sqrt{\beta^2-h^2} \quad (37)$$

$$b = \sqrt{(1+\beta^2) + (\beta+h)\sqrt{1+\beta^2} + \beta h} + \sqrt{(1+\beta^2) - (\beta+h)\sqrt{1+\beta^2} + \beta h} + \sqrt{2\beta(\beta+h)} \quad (38)$$

$$c = \sqrt{(1+\beta^2) + (\beta+h)\sqrt{1+\beta^2} + \beta h} + \sqrt{(1+\beta^2) - (\beta+h)\sqrt{1+\beta^2} + \beta h} - \sqrt{2\beta(\beta+h)} \quad (39)$$

$$d = \sqrt{1+\beta^2-h^2} - \sqrt{\beta^2-h^2} \quad (40)$$

Premultiplying (36) by (33) [neglecting the $1/(1-t^2)$ factors] gives the matrix

$$\begin{bmatrix} 1 & -Z_1 t \\ -t/Z_1 & 1 \end{bmatrix}\begin{bmatrix} at^2+t & bt \\ ct & dt^2+1 \end{bmatrix} = \begin{bmatrix} (a-cZ_1)t^2+1 & -dZ_1t^3+(b-Z_1)t \\ -at^3/Z_1+(c-a/Z_1)t & (d-b/Z_1)t^2+1 \end{bmatrix}. \quad (41)$$

All elements of this matrix are divisible by $(1-t^2)$ giving the two conditions

$$Z_1 = \frac{a+1}{c} = \frac{b}{d+1} \quad (42)$$

which are equivalent, since the determinant of (36) is unity. Eq. (41) now reduces to

$$\begin{bmatrix} 1 & \dfrac{dbt}{d+1} \\ \dfrac{act}{a+1} & 1 \end{bmatrix} \equiv \begin{bmatrix} 1 & Z_2 t \\ t/Z_2 & 1 \end{bmatrix} \quad (43)$$

giving

$$Z_2 = Z_1/c = dZ_1. \quad (44)$$

Eqs. (43) and (44) are the expressions for the normalized even-mode impedances of the two sections of the coupler, and are functions of β and h only, *i.e.*, of the two parameters of the original insertion loss function [(4)–(6)]. In designing a coupler, the coupling, ripple and bandwidth are the quantities of interest, and these are perhaps more conveniently expressed in terms of J and H, themselves functions of β and h, as defined by (14) and (17). The bandwidth, in terms of the electrical length θ of one coupling section, extends from θ_0 to $\pi-\theta_0$, where θ_0 is defined by (15), which may be written in terms of J in the form

$$\frac{1}{\cos\theta_0} = \cosh J/n. \quad (5a)$$

Simple manipulation of the basic (4)–(6) give the following equations for the mean coupling C db and the coupling tolerance on ripple $\pm R$ db:

$$C = 10\log_{10}\frac{\sinh 2H}{\sinh 2J} \quad (45)$$

$$R = 10\log_{10}\frac{\tanh H}{\tanh J}\cdot \quad (46)$$

The bandwidth ratio $(\pi-\theta_0)/\theta_0$ as a function of the coupling tolerance of 3-db and 10-db couplers for a given number of elements is shown in Fig. 3. This graph was obtained from (5a), (45) and (46).

Eqs. (37)–(40), (42) and (44) have been applied to give several two-element couplers, design data for which are as follows:

a) Coupling 10 db ± 0.5 db
Bandwidth 0.5–2.0 Gc/s.

This data is satisfied by $\beta=0.3535$, $h=0.1672$. Direct substitution of these values into the equations gives

$$Z_1 = 1.562 \qquad Z_2 = 1.154.$$

Remembering that Z_1 and Z_2 are normalized even-mode impedances of the two-element coupler, being related to the odd-mode impedances by (2), and renormalizing to 50 ohms, gives the following ohmic values of the impedances in the two sections of the coupler:

Section	1	2
Z_{oe}	78.1	57.7
Z_{oo}	32.0	43.3

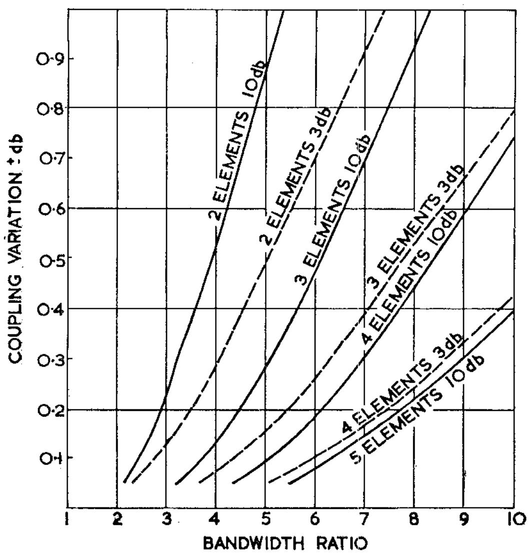

Fig. 3—Coupling tolerance of 3-db and 10-db stepped directional couplers as a function of bandwidth.

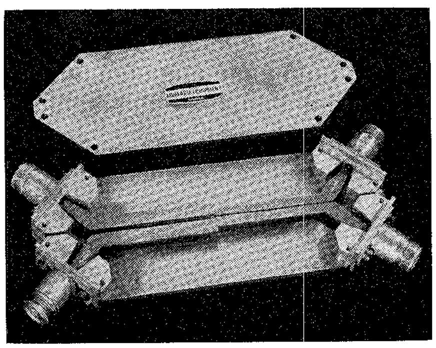

Fig. 4—Two-element coupler, 10 db ± 0.5 db, 0.8–2.0 Gc/s.

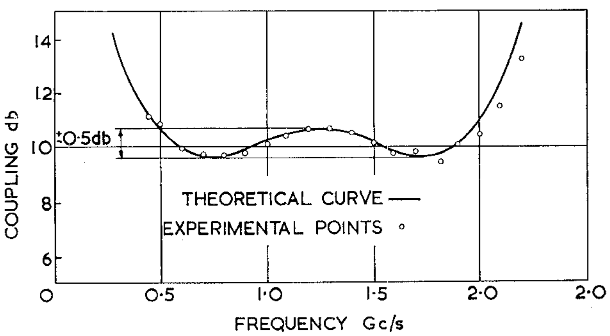

Fig. 5.—Two-element 10-db coupler.

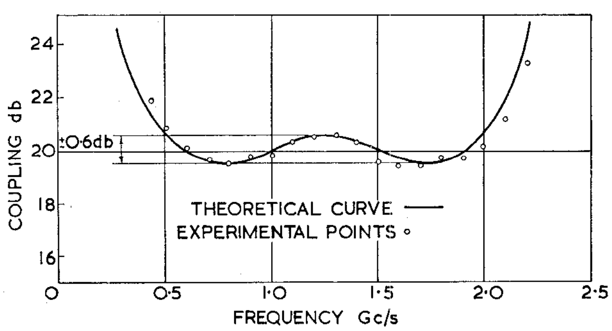

Fig. 6—Two-element 20-db coupler.

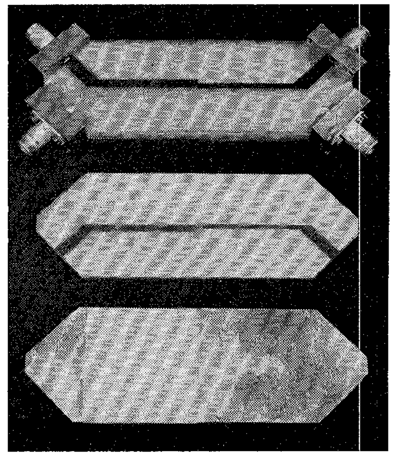

Fig. 7—Two-element coupler using interleaved-strip printed circuit construction.

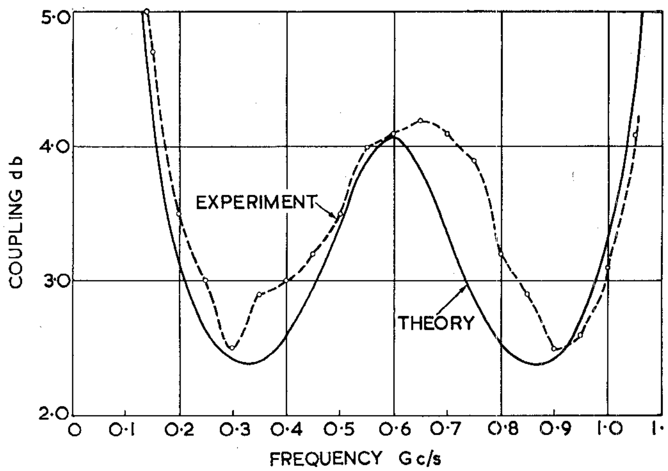

Fig. 8—Two-element hybrid.

This coupler was made in side-coupled stripline using a ground-plane spacing of 5/16 inch and a strip thickness of 1/16 inch. The dimensions were calculated using Getsinger's accurate method,[9] and a photograph of the coupler is shown in Fig. 4. The result for the coupling is given in Fig. 5, which indicates the very good agreement between the experimental results and the theory. The directivity is better than 18 db and the VSWR better than 1.13 over the design frequency band.

b) Coupling 20 db ± 0.6 db
Bandwidth 0.5–2.0 Gc/s
$\beta = 0.1070$, $h = 0.0519$.

The formulas give $Z_1 = 1.143$, $Z_2 = 1.046$, *i.e.*, the following values of the impedances in the coupler:

Section	1	2
Z_{oe}	57.4	52.3
Z_{oo}	43.6	47.8

[9] W. J. Getsinger, "Coupled bars between parallel plates," IRE Trans. on Microwave Theory and Techniques, vol. MTT-10, pp. 65–72; January, 1962.

This coupler is similar in construction to a) and the comparison between theory and experiment for the coupling is shown in Fig. 6. The directivity was better than 14 db and the VSWR better than 1.10 over the frequency band.

c) Coupling 2.4–4.07 db
Bandwidth 0.155–1.035 Gc/s.

This data gives $J=0.8423$, $H=1.200$, or $\beta=1.164$, $h=0.8487$, which upon substitution in the formulas give $Z_1=3.52$, $Z_2=1.692$, *i.e.*,

Section	1	2
Z_{oe}	176	84.6
Z_{oo}	14.2	29.5

The coupler was made in the interleaved printed circuit configuration, as described by Getsinger[10] and shown in Fig. 7. The ground-plane spacing in the more closely coupled section is greater than in the more loosely coupled section in order to give reasonable dimensions, in particular better matching between the dimensions of the inner strips. The comparison between the experimental results and theory is shown in Fig. 8. The directivity is greater than 20 db and the input VSWR better than 1.2 over the band.

The phase properties of these couplers are of interest in that, since they are asymmetric, the phase division of the output voltages is not 90° independent of frequency, but has a marked frequency dependence, as noted by Sweet.[11] The phase division for the two-element couplers is calculated in Appendix III, and the theoretical equations (82)–(85) have been verified by measurements on the two-element coupler c).

A. General Formulas: n=Z

It would be very useful to derive formulas for the Z_r in the general case similar to those found by Takahasi[12] and Green[13] for the low-pass filter. The present problem is more complicated, however, and although it is felt that general formulas do exist, no attempt has yet been made to find them. It is fortunate that only a small number of elements are ever required, for example a 10:1 bandwidth can be obtained for a coupling of 3 db ± 0.43 db with only four elements.

It is quite simple to write down general formulas for the Z_r in the case of a small number of elements, although the expressions are inevitably rather complicated. As an example, take $n=3$. Application of (22) gives

$$\Gamma(t) = \frac{\cosh J\, t(t^2 + T_1 t + |t_1|^2)}{\cosh H\,(t + t_3')(t^2 + T_1' t + |t_1'|^2)} \tag{47}$$

where T_1, T_1', $|t_1|^2$ and $|t_1'|^2$ are obtained from (23) and (24) with the unprimed terms given by substituting J for H. Application of (26) and (31) gives the transfer matrix

$$\begin{bmatrix} a_2 t^2 + 1 & b_3 t^3 + b_1 t \\ c_3 t^3 + c_1 t & d_2 t^2 + 1 \end{bmatrix} \tag{48}$$

where

$$b_3 = h(\cosh H + \cosh J)$$
$$c_3 = h(\cosh H - \cosh J)$$
$$a_2 = h(t_3' + T_1')\cosh H + hT_1 \cosh J$$
$$d_2 = h(t_3' + T_1')\cosh H - hT_1 \cosh J$$
$$b_1 = h(|t_1'|^2 + t_3' T_1') \cosh H + h|t_1|^2 \cosh J$$
$$c_1 = h(|t_1'|^2 + t_3' T_1') \cosh H - h|t_1|^2 \cosh J.$$

In deriving this matrix use has been made of the fact that

$$ht_3' |t_1'|^2 \cosh H = 1 \tag{49}$$

which enables the matrix to be normalized so that the coefficient of t^0 in the even polynomial is always unity. It would appear that an equation similar to (49) may be written down for any value of n. The matrix (48) may be broken down into its constituent matrices to give the general formulas

$$Z_1 = \frac{a_2 + 1}{c_3 + c_1} = \frac{b_3 + b_1}{d_2 + 1} \tag{50}$$

$$Z_2 = \frac{c_3 Z_1 + 1}{c_1 - 1/Z_1} = \frac{b_1 - Z_1}{b_3/Z_1 + 1} \tag{51}$$

$$Z_3 = \frac{b_3 Z_2}{Z_1} = \frac{Z_2}{c_3 Z_1} \cdot \tag{52}$$

It can be shown easily that the alternative forms given in (50)–(52) are equivalent by applying the condition that the determinant of matrix (48) is equal to $(1-t^2)^3$. Similar formulas for $n=4$ and $n=5$ are given in Appendix IV.

Eqs. (50) to (52) have been applied to design a coupler of 20 db ± 0.5 db over a 6:1 band, the relevant constants being $\beta=0.1063$, $h=0.0495$, leading to normalized even-mode impedances of $Z_1=1.164$, $Z_2=1.080$, $Z_3=1.030$, the actual impedance values in ohms in the three sections being as follows:

Section	1	2	3
Z_{oe}	58.2	54.0	51.5
Z_{oo}	42.95	46.3	48.55

[10] W. J. Getsinger, "A coupled stripline configuration using printed-circuit construction that allows very close coupling," IRE Trans. on Microwave Theory and Techniques, vol. MTT-9, pp. 535–544; November, 1961.

[11] O. Sweet, "Analysis of a two-section coupler," IRE Trans. on Microwave Theory and Techniques, vol. MTT-10, p. 295; July, 1962.

[12] L. Weinberg and P. Slepian, "Takahasi's results on Tchebycheff and Butterworth ladder networks," IRE Trans. on Circuit Theory, vol. CT-7, pp. 88–101; June, 1960.

[13] E. Green, "Synthesis of ladder networks to give Butterworth or Chebyshev response in the pass-band," *Proc. I.E.E.*, vol. 101 IV, pp. 192–203; January, 1954.

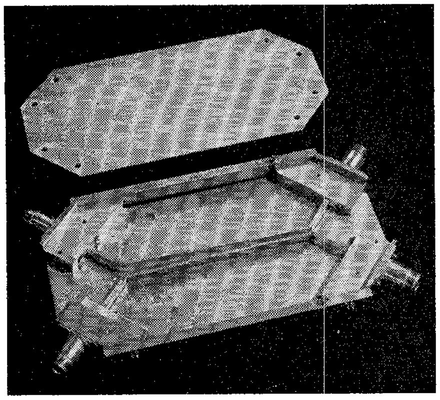

Fig. 9.—Three-element broadside-coupled stripline coupler, 20 db ±0.5 db, 0.5–3.0 Gd/s.

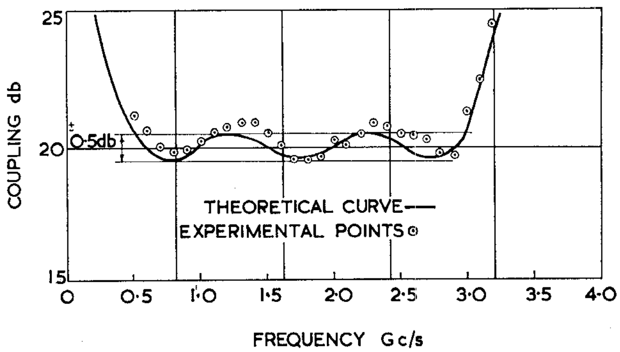

Fig. 10—Three-element 20-db coupler.

The coupler was constructed in broadside-coupled stripline, as shown in Fig. 9, using Cohn's formulas.[14,15] The remarkably good agreement obtained between theory and experiment is shown in Fig. 10. The directivity is better than 20 db and the VSWR better than 1.20 over the band.

III. Practical Disadvantages of Multi-Element Couplers

The main difficulty in applying the theory is the fact that the coupling of the tightest coupled element (considered as a single-element coupler) is always considerably tighter than the actual over-all coupling. This is not a serious drawback for loose couplers, *e.g.*, the coupling corresponding to Z_1 in the above example of a 20-db coupler is 16 db, which is easy to manufacture. In the case of 3-db couplers the tightest coupling may become extremely tight. Preliminary calculations on a four-element 3-db coupler for a bandwidth of 10:1 indicate that the tightest coupling is 1 db, which should be possible to manufacture, however.

Another difficulty is that it is necessary to maintain very close tolerances on all dimensions, and in particular the electrical lengths of the sections must be all equal. Multiple reflections within the coupler and at the ends, where it is joined to the input lines, may also be troublesome.

The problems of input VSWR and directivity are also associated with undesirable internal reflections within the coupler, and in differences in the effective electrical lengths between the even and odd modes. The introduction of dielectric which affects the velocity of one mode with respect to the other in such a way as to cancel the unwanted signal in the isolated arm would appear to be a useful nonempirical way of obtaining good performance.

IV. Conclusions

Having proved that a lumped-element ladder network forms a prototype for a stepped-impedance filter, it has been shown how to synthesize a class of asymmetric multi-element directional couplers. The synthesis leads to explicit formulas for the essential parameters, *i.e.*, the normalized even-mode impedance of each coupler element, as a function of the bandwidth, coupling and ripple. Examples of two- and three element couplers have been made and tested, giving excellent agreement with the theory. It now seems possible to design 3-db couplers for operation over bandwidths of one decade or more. The phase division is highly frequency dependent, but by careful choice of reference planes in the output ports, this can be made approximately 90° over the bandwidth of the coupler.

Appendix I

Equivalence between a Multi-Element Directional Coupler and a Stepped-Impedance Filter

The directional coupler shown in Fig. 1 consists of two identical lines 1–4 and 2–3 with uniform spacing over the electrical length θ. These lines are terminated by input and output ports of characteristic impedance unity. The coupler may be analyzed by the method of Reed and Wheeler,[16] which also gives a good physical picture of the directional coupler operation. The coupled lines may be analyzed in two modes known as the even and odd modes.[2,3] In the even mode, Fig. 11(a), in-phase signals are applied to ports 1 and 2 and a voltage maximum occurs on the line of symmetry. A magnetic wall may be located at the symmetry plane without affecting the field distribution in this mode. The odd mode is shown in Fig. 11(b), when out-of-phase signals are applied to ports 1 and 2 and a voltage zero occurs on the line of symmetry. Hence an electric wall (short circuit) may be located without affecting the operation. The electric field distribution for the two

[14] S. B. Cohn, "Characteristic impedances of broadside-coupled strip transmission lines," IRE Trans. on Microwave Theory and Techniques, vol. MTT-8, pp. 633–637; November, 1960.

[15] S. B. Cohn, "Thickness corrections for capacitive obstacles and strip conductors," IRE Trans. on Microwave Theory and Techniques, vol. MTT-8, pp. 638–644; November, 1960.

[16] J. Reed and G. J. Wheeler, "A method of analysis of symmetrical four port networks," IRE Trans. on Microwave Theory and Techniques, vol. MTT-4, pp. 246–252; October, 1956.

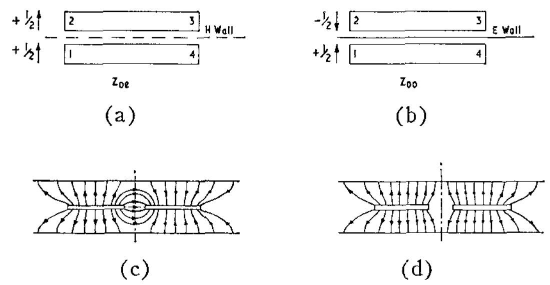

Fig. 11—Reed and Wheeler directional coupler theory.

modes is shown in Figs. 11(c) and 11(d) for the case of coupled striplines. In each case, the analysis reduces to that of a two-port network and, by superposition, the sum of the two cases is equivalent to a single signal of unit amplitude applied to port 1.

The amplitude and phase of the signals emerging from the four ports are[16]

$$
\begin{aligned}
A_1 &= \tfrac{1}{2}(\Gamma_{oe} + \Gamma_{oo}) \\
A_2 &= \tfrac{1}{2}(\Gamma_{oe} - \Gamma_{oo}) \\
A_3 &= \tfrac{1}{2}(T_{oe} - T_{oo}) \\
A_4 &= \tfrac{1}{2}(T_{oe} + T_{oo})
\end{aligned}
\tag{53}
$$

where Γ_{oe} and Γ_{oo} are the reflected waves, and T_{oe} and T_{oo} are the transmitted waves, for the even and odd two-port networks, respectively. The analysis of the two-port networks is most easily carried out by use of the transfer matrix of each, remembering the following results:

$$
\Gamma = \frac{A + B - C - D}{A + B + C + D} \tag{54}
$$

$$
T = \frac{2}{A + B + C + D} \tag{55}
$$

where Γ and T are the reflected and transmitted waves with a matched load on the output port, and where the matrix elements are normalized with respect to this load impedance. In the case of the coupled transmission lines, the transfer matrix for the even mode is

$$
\begin{bmatrix} \cos\theta & jZ_{oe}\sin\theta \\ (j\sin\theta)/Z_{oe} & \cos\theta \end{bmatrix} \tag{56}
$$

and for the odd mode it is

$$
\begin{bmatrix} \cos\theta & jZ_{oo}\sin\theta \\ (j\sin\theta)/Z_{oo} & \cos\theta \end{bmatrix}. \tag{57}
$$

Application of (54) and (55) to these transfer matrices gives

$$
\begin{aligned}
\Gamma_{oe} &= \frac{j(Z_{oe} - 1/Z_{oe})\sin\theta}{2\cos\theta + j(Z_{oe} + 1/Z_{oe})\sin\theta} \\
\Gamma_{oo} &= \frac{j(Z_{oo} - 1/Z_{oo})\sin\theta}{2\cos\theta + j(Z_{oo} + 1/Z_{oo})\sin\theta} \\
T_{oe} &= \frac{2}{2\cos\theta + j(Z_{oe} + 1/Z_{oe})\sin\theta} \\
T_{oo} &= \frac{2}{2\cos\theta + j(Z_{oo} + 1/Z_{oo})\sin\theta}.
\end{aligned}
\tag{58}
$$

When the condition

$$
Z_{oe}Z_{oo} = 1 \tag{2}
$$

is satisfied (58) reduces as follows:

$$
\begin{aligned}
\Gamma_{oe} = -\Gamma_{oo} &= \frac{j(Z_{oe} - 1/Z_{oe})\sin\theta}{2\cos\theta + j(Z_{oe} + 1/Z_{oe})\sin\theta} \\
T_{oe} = T_{oo} &= \frac{2}{2\cos\theta + j(Z_{oe} + 1/Z_{oe})\sin\theta}.
\end{aligned}
\tag{59}
$$

From (53) the signals emerging from the four ports of the directional coupler are then:

$$
A_1 = 0 \tag{60}
$$

$$
A_2 = \Gamma_{oe} = \frac{j(Z_{oe} - 1/Z_{oc})\sin\theta}{2\cos\theta + j(Z_{oe} + 1/Z_{oe})\sin\theta} \tag{61}
$$

$$
A_3 = 0 \tag{62}
$$

$$
A_4 = T_{oe} = \frac{2}{2\cos\theta + j(Z_{oe} + 1/Z_{oe})\sin\theta}. \tag{63}
$$

Eqs. (60) and (62) show that the coupler is perfectly matched and isolated at all frequencies, while the coupling to ports 2 and 4 vary with frequency.

The proof that the analysis of the directional coupler is equivalent to that of a stepped impedance filter [providing (2) holds] is now evident from (61) and (63). These are simply the reflection and transmission coefficients respectively of a uniform transmission line of electrical length θ and characteristic impedance Z_{oe}, normalized to its input and output terminations. It is a trivial extension to show that the multi-element coupler retains the perfect VSWR and isolation property if each element has the same effective characteristic impedance, *i.e.*, if

$$
Z_{oe1}Z_{oo1} = Z_{oe2}Z_{oo2} = \cdots = Z_{oen}Z_{oon} = 1 \tag{64}
$$

and hence the analysis of this stepped coupler is identical to that of a multi-element stepped filter, as shown in Fig. 2.

The simple formulas for the power insertion loss between arms 1 and 4 of the directional coupler, *e.g.*, (1) and (3), are given by application of the well-known formula

$$
L = 1 + \tfrac{1}{4}(A - D)^2 - \tfrac{1}{4}(B - C)^2 \tag{65}
$$

where A and D are the real and B, C the imaginary elements of the transfer matrix for a lossless network.

Appendix II

General Synthesis of Transmission-Line Stepped-Impedance Filters

Riblet[8] has proved that a rational function of $t=\tanh\theta$ (in Riblet's notation $t=1/p$) with real coefficients written in the form

$$Z(t) = \frac{A(t) + B(t)}{D(t) + C(t)} \tag{29}$$

with A and D even and B and C odd functions of t, is the input impedance of a stepped-impedance filter with equal line lengths θ if

1) Z is a positive real function of t, and

2) $A(t)D(t) - B(t)C(t) = C'(1-t^2)^n \quad (66)$

where C' is a constant.

Leo Young[17] has pointed out that a third condition is required, namely that

3) The degrees of the numerator and denominator in (29) are the same (otherwise a stub will be required).

Riblet's theorem may be extended by noting that any two-port lumped-element ladder network consisting of simple series reactances and shunt susceptances terminated by pure resistances has a transfer matrix of the form

$$\begin{bmatrix} A(p) & B(p) \\ C(p) & D(p) \end{bmatrix} \tag{67}$$

where p is the complex frequency variable, $A(p)$, $D(p)$ are even functions of p, and $B(p)$, $C(p)$ are odd functions of p. The insertion loss is therefore

$$L = 1 + \tfrac{1}{4}(A-D)^2 - \tfrac{1}{4}(B-C)^2 \tag{68}$$

and the square of the magnitude of the input reflection coefficient is

$$|\Gamma(p)|^2 = \frac{\tfrac{1}{4}(A-D)^2 - \tfrac{1}{4}(B-C)^2}{1 + \tfrac{1}{4}(A-D)^2 - \tfrac{1}{4}(B-C)^2}. \tag{69}$$

Now if the frequency variable is replaced by a linear function in $\cos\theta$ and $\sin\theta$, then it is easily shown, by multiplying the numerator and denominator of (69) by

$$\frac{1}{\cos^{2n}\theta} = (1-t^2)^n,$$

that

$$|\Gamma(t)|^2 = \frac{f_1(t^2) + f_2(t^2)}{(1-t^2)^n + f_1(t^2) + f_2(t^2)}. \tag{70}$$

[17] L. Young, "Concerning Riblet's theorem," IRE Trans. on Microwave Theory and Techniques, vol. MTT-7, pp. 477–478; October, 1959.

Richards[7] has shown that the right half of the p plane maps into the right half of the t plane, and since $Z(p)$ is positive real therefore $Z(t)$ must be positive real, proving that condition 1) holds. The impedance $Z(t)$ obtained from (70) will give (29). Since $Z(t)$ is positive real, all the coefficients appearing in Z are of the same sign, and the degrees of the numerator and denominator differ at most by unity. However, in the case of a prototype ladder network where the poles of the insertion loss function are all at infinity, this difference is zero. The reflection coefficient is

$$\Gamma(t) = \frac{(A-D) + (B-C)}{(A+D) + (B+C)} \tag{71}$$

giving

$$|\Gamma(t)|^2 = \frac{(A-D)^2 - (B-C)^n}{(A+D)^2 - (B+C)^2}. \tag{72}$$

Eqs. (70) and (72) are identical, and equating the differences between the numerator and denominator of each expression for $\Gamma(t)^2$ gives

$$A(t)\cdot D(t) - B(t)\cdot C(t) = \tfrac{1}{4}(1-t^2)^n$$

which is (66), *i.e.*, proving condition 2). The following theorem may now be stated:

An insertion loss function of the form

$$L = 1 + [f_1(a\cos\theta + b\sin\theta)]^2 + [f_2(a\cos\theta + b\sin\theta)]^2 \tag{73}$$

can be realized as a stepped impedance filter with real positive characteristic impedances if the function

$$L = 1 + [f_1(\omega)]^2 + [f_2(\omega)]^2 \tag{74}$$

having all its poles at infinity, is realizable as a two-port ladder network consisting of simple lossless series reactances and shunt susceptances terminated by resistances.

The restriction that the impedance function shall have a numerator and denominator of equal degree is implied by the form of (74), since this has all its poles at infinity, and therefore there is no need to state the restriction explicitly in the theorem.

The prototype function considered in the present context is given by

$$L = 1 + \beta^2 - h^2T_n^2(\omega) \tag{75}$$

and it is quickly proved that this satisfies the requirements of the above theorem. The only difference between (75) and the usual Chebyshev insertion loss function encountered in ladder network filter theory is in the sign of the $h^2T_n^2(\omega)$ term. Eq. (75) can still be physically realized however, since the roots of $|\Gamma(j\omega)|^2$ formed from it are similar, giving

$$|\Gamma(j\omega)|^2 = \Gamma(j\omega)\cdot\Gamma(-j\omega) \tag{76}$$

with the roots of $\Gamma(j\omega)$ occurring as complex conjugate pairs. Hence $\Gamma(p)$ (where $p=\sigma+j\omega$) can be selected to be analytic in the right-half plane, *i.e.*, $Z(p)$ is positive real, a necessary and sufficient condition for the physical realization of the insertion loss function of (75). Application of the above theorem shows that the transformation $\omega \rightarrow \cos\theta/\cos\theta_0$ in (75), giving (6), enables the resulting insertion loss function to be realized as a stepped impedance filter.

An alternative and possibly more direct proof that (6) can be synthesized as a stepped impedance filter is obtained by applying the reasoning which proved the theorem of this appendix, and showing that (66) is true. The main text shows that $Z(t)$ is positive real, and (26) that the numerator and denominator of $Z(t)$ are of the same degree.

Appendix III

Phase Properties of the Asymmetric Coupler

The couplings to arms 2 and 4 of the asymmetric coupler are given by Γ and T (see Fig. 2) derived from the transfer matrix (7), *i.e.*,

$$\Gamma = \frac{(A_n - D_n) + j(B_n - C_n)}{(A_n + D_n) + j(B_n + C_n)} \tag{77}$$

$$T = \frac{2}{(A_n + D_n) + j(B_n + C_n)} \tag{78}$$

and the power division between arms 2 and 4 is

$$\eta = \frac{\Gamma}{T} = \frac{(A_n - D_n) + j(B_n - C_n)}{2}. \tag{79}$$

In the case of a symmetric coupler, *e.g.*, the three-quarter wavelength coupler described previously,[3] the phase difference between Γ and T is 90° at all frequencies because $A_n = D_n$. This condition does not hold for an asymmetric coupler, and the phase difference is

$$\phi = \tan^{-1}\frac{B_n - C_n}{A_n - D_n}. \tag{80}$$

Thus, while the symmetric 3-db coupler can be used as a 90° hybrid, this is not immediately true of the asymmetric 3-db coupler, which has a phase shift ϕ varying in an approximately linear fashion over the operating frequency band of the coupler. However, by the addition of a length of line to one of the output arms, this phase variation can be approximately cancelled, and a usable 90° hybrid performance obtained. In other words, reference planes exist in arms 2 and 4 where the phases differ by approximately 90° over the band.

As an example take the case of the two element asymmetric coupler. This has a transfer matrix given by (36) to (40), but in this simple case it is preferable to form the matrix directly by analysis, giving

$$\begin{bmatrix} \cos^2\theta - (Z_1\sin^2\theta)/Z_2 & j(Z_1+Z_2)\sin\theta\cos\theta \\ j(1/Z_1 + 1/Z_2)\sin\theta\cos\theta & \cos^2\theta - (Z_2\sin^2\theta)/Z_1 \end{bmatrix}. \tag{81}$$

The phase difference is given by (54), *i.e.*,

$$\phi = \tan^{-1}(K_1\cot\theta) \tag{82}$$

where

$$K_1 = \frac{(Z_1 + Z_2) - (1/Z_1 + 1/Z_2)}{(Z_2/Z_1 - Z_1/Z_2)}. \tag{83}$$

At midband $\theta = \pi/2$ giving $\phi = 0$. At zero frequency $\phi = \pi/2$, and at $\theta = \pi$, $\phi = -\pi/2$, so that by adding a line of electrical length $\pi/2$ at midband to ϕ the phase will be exactly $\pi/2$ at these three points. In order to decide on which arm to include the extra line it is necessary to find the exact meaning of the sign of the phase. Negative phase implies a forward traveling wave, *i.e.*, one emerging from an output arm, so by writing

$$\frac{\Gamma}{T} = \frac{|\Gamma|}{|T|}\frac{e^{-j\theta_2}}{e^{-j\theta_4}} = \frac{|\Gamma|}{|T|}e^{j\phi}$$

then

$$\phi = \theta_4 - \theta_2. \tag{84}$$

Hence, the extra line must be added to arm 4, which is the one leading into the closely coupled element. The phase difference between the new reference planes is now

$$\psi = \phi + \theta = \tan^{-1}(K_1\cot\theta) + \theta \tag{85}$$

where in the range $0 < \theta < \pi$

$$-\frac{\pi}{2} < \tan^{-1}(K_1\cot\theta) < +\frac{\pi}{2}. \tag{86}$$

Maximum and minimum values of ψ occur when

$$\frac{d\psi}{d\theta} = 0, \qquad i.e., \text{ for } \theta_m = \tan^{-1} \pm \sqrt{K_1} \tag{87}$$

and these values of θ substituted in (85) give the points of maximum deviation of ψ from 90°. For example, in the case of a two-element coupler designed to give a coupling of 3 db ± 0.9 db over a 7:1 band then $K_1 = 2.8$, giving

$$\theta_m = 59.3° \text{ and } 110.7°.$$

Substituting in (85) gives

$$= 118.6° \text{ and } 61.4° \text{ (respectively)}$$

i.e., a phase variation of $\pm 28.6°$ about 90°.

When the coupling ripple level is decreased, the phase linearity is improved, *e.g.*, for a 5:1 bandwidth giving ± 0.48 db ripple then the phase variation reduces to $\pm 22.2.°$

Variations of this order of magnitude are quite acceptable in many applications. In cases where the phase is required to be much more exactly equal to 90°, then the symmetrical form of the coupler should be used.

Appendix IV
General Formulas

A. $n=4$

Remembering (20), (25) becomes

$$\Gamma(t) = \frac{\sinh J}{\sinh H} \frac{t(t+t_2)(t^2+T_1t+|t_1|^2)}{(t+t_2')(t+t_4')(t^2+T_1't+|t_1'|^2)}$$

giving the transfer matrix

$$\begin{bmatrix} a_4t^4+a_2t^2+1 & b_3t^3+b_1t \\ c_3t^3+c_1t & d_4t^4+d_2t^2+1 \end{bmatrix} \quad (88)$$

where

$$\left.\begin{matrix} a_4 \\ d_4 \end{matrix}\right\} = h(\sinh H \pm \sinh J)$$

$$\left.\begin{matrix} b_3 \\ c_3 \end{matrix}\right\} = h(T_1'+t_2'+t_4')\sinh H \pm h(T_1+T_2)\sinh J$$

$$\left.\begin{matrix} a_2 \\ d_2 \end{matrix}\right\} = \begin{matrix} h[|t_1'|^2+T_1'(t_2'+t_4')+t_2't_4']\sinh H \\ \pm h[|t_1|^2+T_1t_2]\sinh J \end{matrix}$$

$$\left.\begin{matrix} b_1 \\ c_1 \end{matrix}\right\} = \begin{matrix} h[T_1't_2't_4'+|t_1'|^2(t_2'+t_4')]\sinh H \\ \pm h|t_1|^2 t_2 \sinh J. \end{matrix}$$

In the above equations the plus sign refers to the upper quantities on the left-hand side, and the negative sign to the lower quantities. The roots $|t_1'|^2$, T_1', etc., are obtained from (23), (24), *et seq.*, with $n=4$. In deriving the equations a relation similar to that of (49) has been used, namely that

$$h|t_1'|^2 t_2't_4' \sinh H = 1 \quad (89)$$

which is readily proved.

Expressions for the four normalized even-mode impedances are now obtained by breaking up matrix (88) into its constituent matrices, giving

$$Z_1 = \frac{a_4+a_2+1}{c_3+c_1} = \frac{b_3+b_1}{d_4+d_2+1} \quad (90)$$

$$Z_2 = \frac{c_3Z_1-a_4+1}{a_4/Z_1-1/Z_1+c_1} = \frac{d_4Z_1-Z_1+b_1}{b_3/Z_1-d_4+1} \quad (91)$$

$$Z_3 = \frac{a_4Z_2/Z_1+1}{c_1-1/Z_1-1/Z_2} = \frac{b_1-Z_1-Z_2}{d_4Z_1/Z_2+1} \quad (92)$$

$$Z_4 = \frac{Z_1Z_3}{a_4Z_2} = \frac{d_4Z_1Z_3}{Z_2}. \quad (93)$$

B. $n=5$

Eq. (22) gives

$$\Gamma(t) = \frac{\cosh J}{\cosh H} \cdot \frac{t(t^2+T_1t+|t_1|^2)(t^2+T_2t+|t_2|^2)}{(t+t_5')(t^2+T_1't+|t_1'|^2)(t^2+T_2't+|t_2'|^2)} \quad (94)$$

leading to the transfer matrix

$$\begin{bmatrix} a_4t^4+a_2t^2+1 & b_5t_5+b_3t^3 \\ c_5t^5+c_3t^3+c_1t & d_4t^4+d_2t^2+1 \end{bmatrix} \quad (95)$$

where

$$\left.\begin{matrix} b_5 \\ c_5 \end{matrix}\right\} = h(\cosh H \pm \cosh J)$$

$$\left.\begin{matrix} a_4 \\ d_4 \end{matrix}\right\} = h(T_1'+T_2'+t_5')\cosh H \pm h(T_1+T_2)\cosh J$$

$$\left.\begin{matrix} b_3 \\ c_3 \end{matrix}\right\} = \begin{matrix} h[|t_1'|^2+|t_2'|^2+T_1'T_2'+t_5'(T_1'+T_2')]\cosh H \\ \pm [|t_1|^2+|t_2|^2+T_1T_2]\cosh J \end{matrix}$$

$$\left.\begin{matrix} a_2 \\ d_2 \end{matrix}\right\} = \begin{matrix} h[T_1'|t_2'|^2+T_2'|t_1'|^2+t_5'(t_1'^2+|t_2'|^2 \\ +T_1'T_2')]\cosh H \pm h[T_1|t_2|^2+T_2|t_1|^2]\cosh J \end{matrix}$$

$$\left.\begin{matrix} b_1 \\ c_1 \end{matrix}\right\} = \begin{matrix} h[|t_1'|^2|t_2'|^2+t_5'(T_1'|t_2'|^2+T_2'|t_1'|^2)]\cosh H \\ \pm h|t_1|^2|t_2|^2\cosh J, \end{matrix}$$

and the normalized even mode impedances are

$$Z_1 = \frac{a_4+a_2+1}{c_5+c_3+1} = \frac{b_5+b_3+b_1}{d_4+d_2+1} \quad (96)$$

$$Z_2 = \frac{(c_5-c_1)Z_1+a_2}{(a_4-1)/Z_1-c_5+c_1} = \frac{(d_4-1)Z_1+b_5+b_1}{(b_5-b_1)/Z_1+d_2} \quad (97)$$

$$Z_3 = \frac{a_4Z_2/Z_1-c_5(Z_1+Z_2)+1}{c_5Z_1/Z_2+c_1-(1/Z_1+1/Z_2)} = \frac{b_5Z_2/Z_1+b_1-(Z_1+Z_2)}{d_4Z_1/Z_2-b_5(Z_1+Z_2)+1} \quad (98)$$

$$Z_4 = \frac{c_5Z_1Z_3/Z_2+1}{c_1-(1/Z_1+1/Z_2+1/Z_3)} = \frac{b_1-(Z_1+Z_2+Z_3)}{b_5Z_2/Z_1Z_3+1} \quad (99)$$

$$Z_5 = \frac{Z_2Z_4}{Z_1Z_3c_5} = \frac{Z_2Z_4b_5}{Z_1Z_3}. \quad (100)$$

Although at first glance the general formulas seem complicated, they are quite simple to apply and would be particularly appropriate for a computer program. They have a good deal of symmetry, which, as suggested in the main text, might imply the existence of general formulas for any value of n.

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